Let

$$L[V_m] - \alpha_m^2 V_m = 0 \tag{16}$$

Then, in view of Eqs. (16) and (A16), Eq. (15) assumes the form

$$\sum_{n=1}^{\infty} [\ddot{T}_{nm}(t) + \omega_{nm}^2 T_{nm}(t)] R_{nm}(r) = -\ddot{V}_m - C_p^2 \bar{F} \tag{17}$$

If we substitute Eq. (13) into the boundary conditions (10) and (11), we find, using Eq. (14),

$$r(\partial V_m/\partial r) - V_m = S_m(r,t), \text{ at } r = a \tag{18}$$

$$r\left(\frac{\partial V_m}{\partial r}\right) - \left[1 - rh\alpha_m^2\left(\frac{G_s}{G_p}\right)\right] V_m + \sum_{n=1}^{\infty} rh\left(\frac{\rho_s}{G_p}\right) R_{nm}\ddot{T}_{nm} + \sum_{n=1}^{\infty}\left\{rR'_{nm} - \left[1 - rh\alpha_m^2\left(\frac{G_s}{G_p}\right)\right] R_{nm}\right\} T_{nm} = -rh\left(\frac{\rho_s}{G_p}\right) V_m + U_m \tag{19}$$

at $r = b$. To eliminate $\ddot{T}_{nm}$ from Eq. (19), we set $r = b$ in Eq. (17) and write

$$\sum_{n=1}^{\infty} R_{nm}\ddot{T}_{nm} = -\sum_{n=1}^{\infty} \omega_{nm}^2 R_{nm} T_{nm} - \ddot{V}_m - C_p^2\bar{F} \tag{20}$$

Upon substitution of Eq. (20) into Eq. (19), we obtain, by virtue of the second condition in Eq. (14)

$$r(\partial V_m/\partial r) - [1 - rh\alpha_m^2(G_s/G_p)]V_m = U_m^*(r,t) \quad \text{at } r = b \tag{21}$$

where

$$U_m^*(r,t) = rh(\rho_s/\rho_p)\bar{F}(b;m;t) + U_m(r,t)$$

The solution of Eq. (16) is

$$V_m(r,t) = A_m^{(1)}(t) I_1(\alpha_m r) + A_m^{(2)}(t) K_1(\alpha_m r) \tag{22}$$

where I_1 and K_1 are modified Bessel functions of order unity of the first and second kinds, respectively. From the boundary conditions (18) and (21), we find $A_m^{(j)}(t) = C_m^{(j)}(t)/\delta_m, j = 1,2$, with

$$C_m^{(1)}(t) = bS_m(a,t)[K_2(\alpha_m b) - h\alpha_m(G_s/G_p)K_1(\alpha_m b)] - aU_m^*(b,t)K_2(\alpha_m a)$$

$$C_m^{(2)}(t) = aU_m^*(b,t)I_2(\alpha_m a) - bS_m(a,t)[I_2(\alpha_m b) + h\alpha_m(G_s/G_p)I_1(\alpha_m b)]$$

$$\delta_m = ab\alpha_m\{I_2(\alpha_m a)[K_2(\alpha_m b) - h\alpha_m(G_s/G_p)K_1(\alpha_m b)] - K_2(\alpha_m a)[I_2(\alpha_m b) + h\alpha_m(G_s/G_p)I_1(\alpha_m b)]\}$$

Finally, the initial conditions in Eq. (12) may be written as

$$\sum_{n=1}^{\infty} R_{nm}(r)T_{nm}(0) = -V_m(r,0) + \frac{(-1)^m f(r)g(0)}{\alpha_m}.$$
$$\sum_{n=1}^{\infty} R_{nm}(r)\dot{T}_{nm}(0) = -\dot{V}_m(r,0) + \frac{(-1)^m f(r)\dot{g}(0)}{\alpha_m} \tag{23}$$

If we apply the orthogonality condition (A54) to Eqs. (17) and (23), we obtain the following initial value problem:

$$\ddot{T}_{nm}(t) + \omega_{nm}^2 T_{nm}(t) = \mathcal{R}_{nm}(t)$$
$$T_{nm}(0) = -\bar{V}_m(n;0) + (-1)^m \bar{f}(n)g(0)/\alpha_m, \tag{24}$$
$$\dot{T}_{nm}(0) = -\dot{\bar{V}}(n;0) + (-1)^m \bar{f}(n)\dot{g}(0)/\alpha_m$$

where

$$\mathcal{R}_{nm}(t) = (R_{nm}, -\ddot{V}_m - C_p^2\bar{F})/(R_{nm},R_{nm})$$

$$\bar{V}_m(n;t) = (R_{nm},V_m)/(R_{nm},R_{nm}), \quad \bar{f}(n) = (R_{nm},f)/(R_{nm},R_{nm})$$

using the notation of Ref. 1 for the right sides of these last two equations.

The solution of the initial value problem in Eq. (24) is

$$T_{nm}(t) = T_{nm}(0)\cos\omega_{nm}t + \left[\frac{\dot{T}_{nm}(0)}{\omega_{nm}}\right]\sin\omega_{nm}t + \left(\frac{1}{\omega_{nm}}\right)\int_0^t \mathcal{R}_{nm}(\tau)\sin\omega_{nm}(t-\tau)d\tau \tag{25}$$

From Eqs. (A58, 8, and 13), it now follows that the circumferential displacement $v(r,z,t)$ is given by

$$v(r,z,t) = \left(\frac{z}{l}\right)f(r)g(t) + \left(\frac{2}{l}\right)\sum_{n=1}^{\infty} V_m(r,t)\sin\alpha_m z + \left(\frac{2}{l}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{nm}(r)T_{nm}(t)\sin\alpha_m z \tag{26}$$

where $f(r)$ and $g(t)$ are prescribed functions, $V_m(r,t)$ is given in Eq. (22), the radial eigenfunctions $R_{nm}(r)$ appear in Eq. (A45), and the generalized coordinates $T_{nm}(t)$ are obtained from Eqs. (25).

Equation (26), then, is a formal representation of the solution of the boundary value problem posed in Ref. 1; it satisfies the equation of motion, all boundary conditions, and all initial conditions.

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Reply by Author to G. L. Anderson

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ANDERSON is correct in pointing out that Eqs. (A58) and (A63) strictly satisfy neither the condition of vanishing shear stress on the inner curved surface, Eq. (A13), nor the boundary condition at the cylinder-casing interface, Eq. (A12). He is also correct when he writes that in many practical applications of the structure considered in Ref. 1, it may be argued that the stiffness term $G_s h(\partial^2 v/\partial z^2)$ in the boundary condition Eq.(A12) dominates the inertia term $\rho_s h(\partial^2 v/\partial t^2)$. Thus, for a structure such as an encased solid propellant grain, which was discussed in Ref. 1, the effect of the inertia term may generally be ignored.

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